Section 16.1: Vector Fields

What We'll Learn In Section 16.1

1. What is a vector field?
2. The gradient vector field
3. Conservative vector fields

1. What is a vector field?

Def:

1. Let D be a set of points in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\vec{F}(x, y)$.

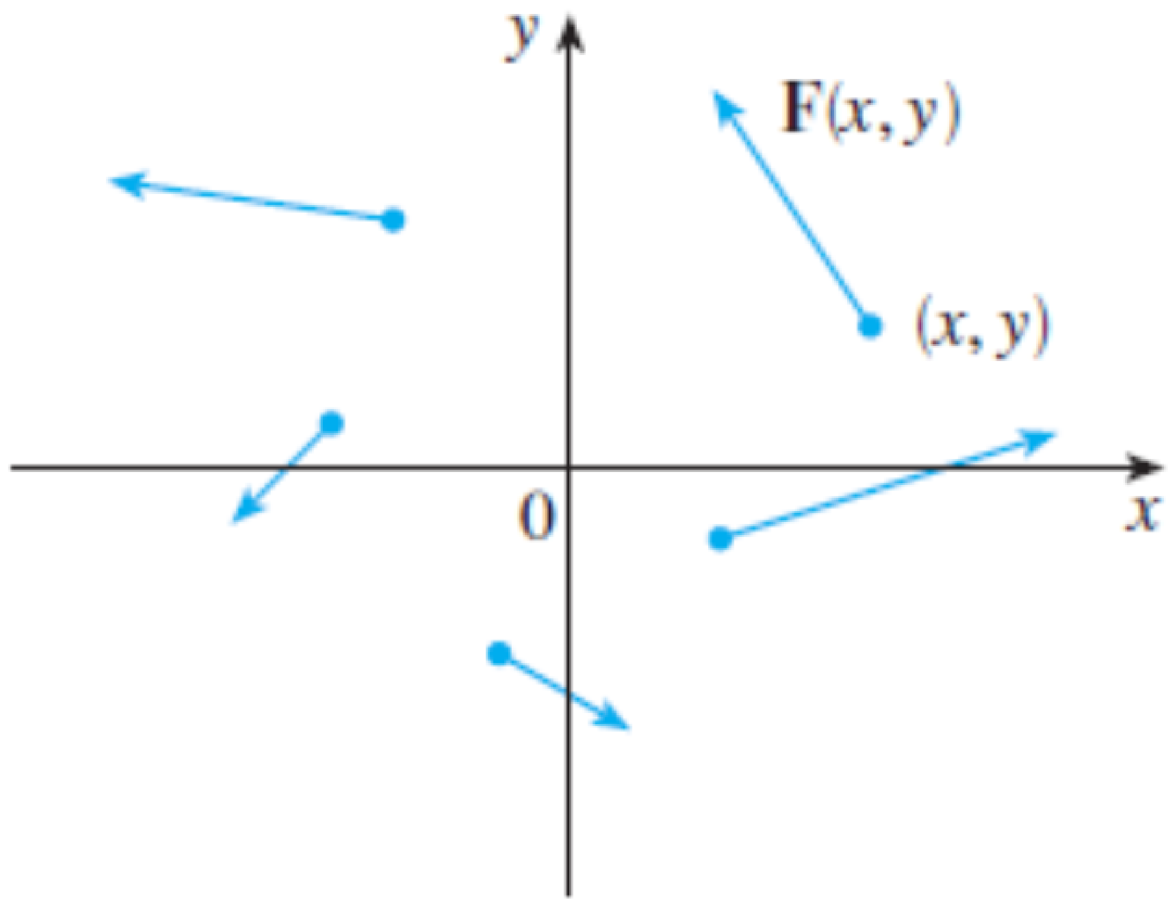
   That is, $\vec{F}: D \subset \mathbb{R}^2 \to V_2$.

Notes:

- We draw a vector field by…
  1) Pick a point in D
  2) Plug that point into $\vec{F}$
  3) Draw the output vector with its tail at the point
  4) Do this for many points in D

1. What is a vector field?

Vector field on $\mathbb{R}^2$



1. What is a vector field?

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   That is, $\vec{F}: D \subset \mathbb{R}^2 \to V_2$.

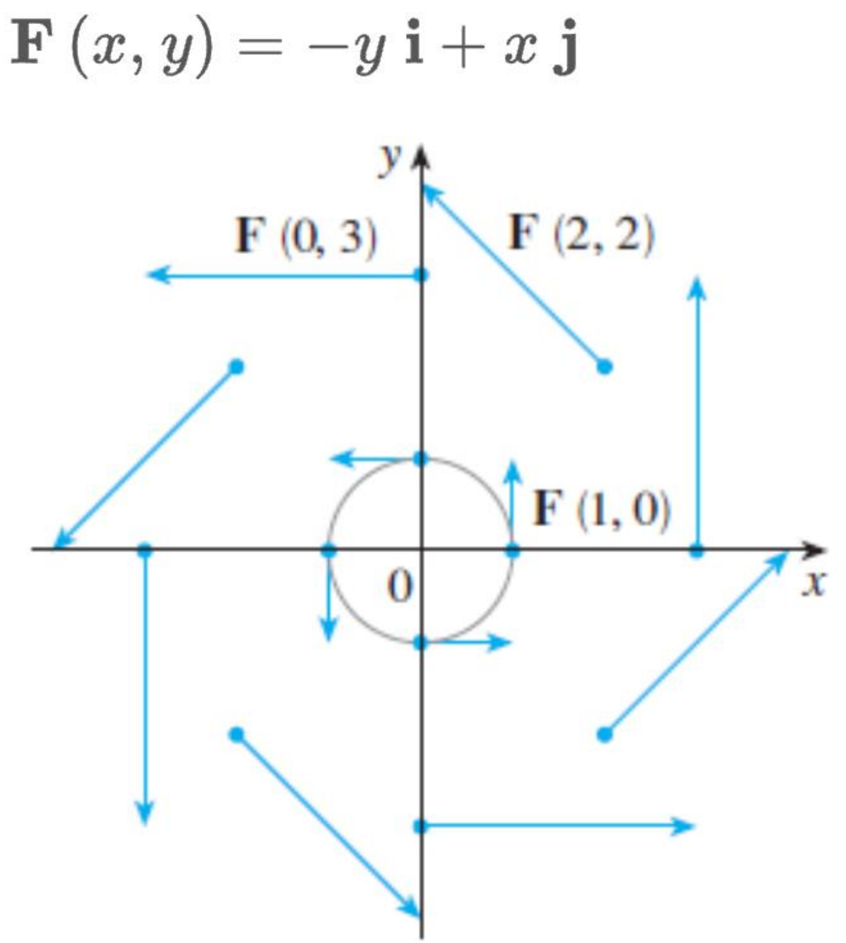
Notes:

- A 2-variable function from D to $\mathbb{R}$ is called a scalar function. We've seen these before.
- If P and Q are to scalar functions, then a vector field on $\mathbb{R}^2$ is the function $\vec{F}$ defined by

$$\vec{F}(x, y) = < P, Q > \text{ or } \vec{F}(x, y) = P\vec{\imath} + Q\vec{\jmath}$$

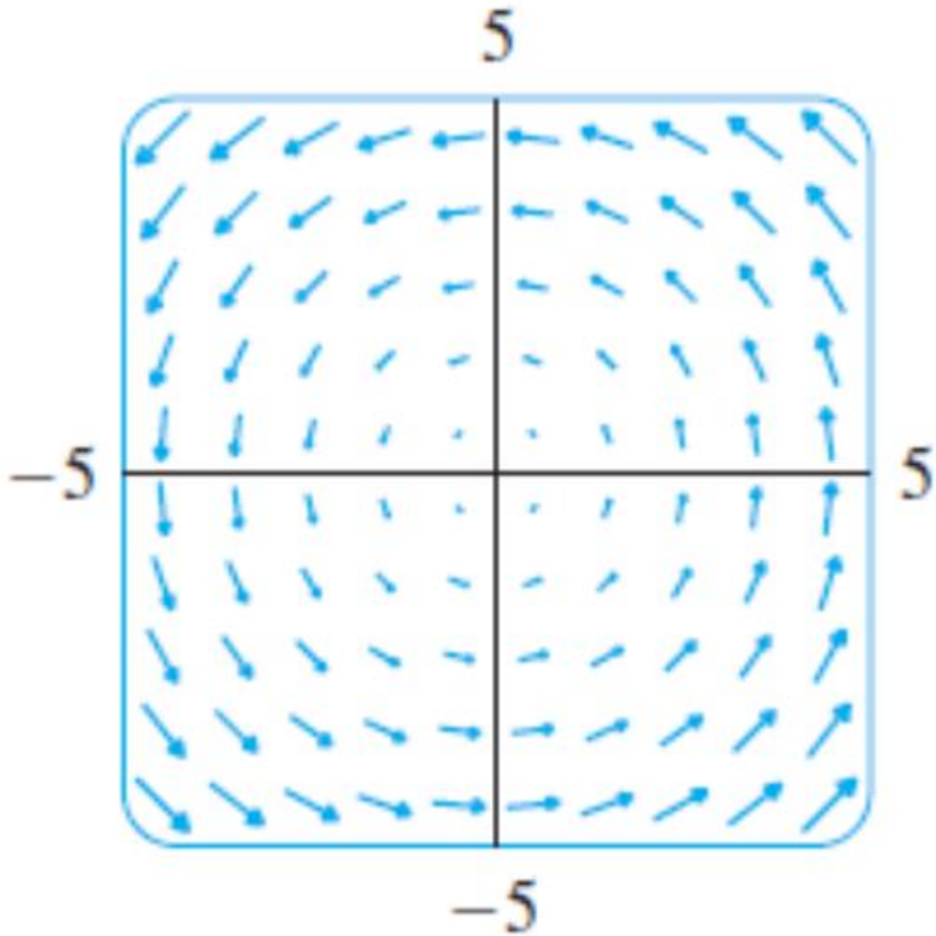
1. What is a vector field?

Ex 1: A vector field on $\mathbb{R}^2$ is defined by $\vec{F}(x, y) = -y\vec{\imath} + x\vec{\jmath}$. Sketch a portion of this vector field.



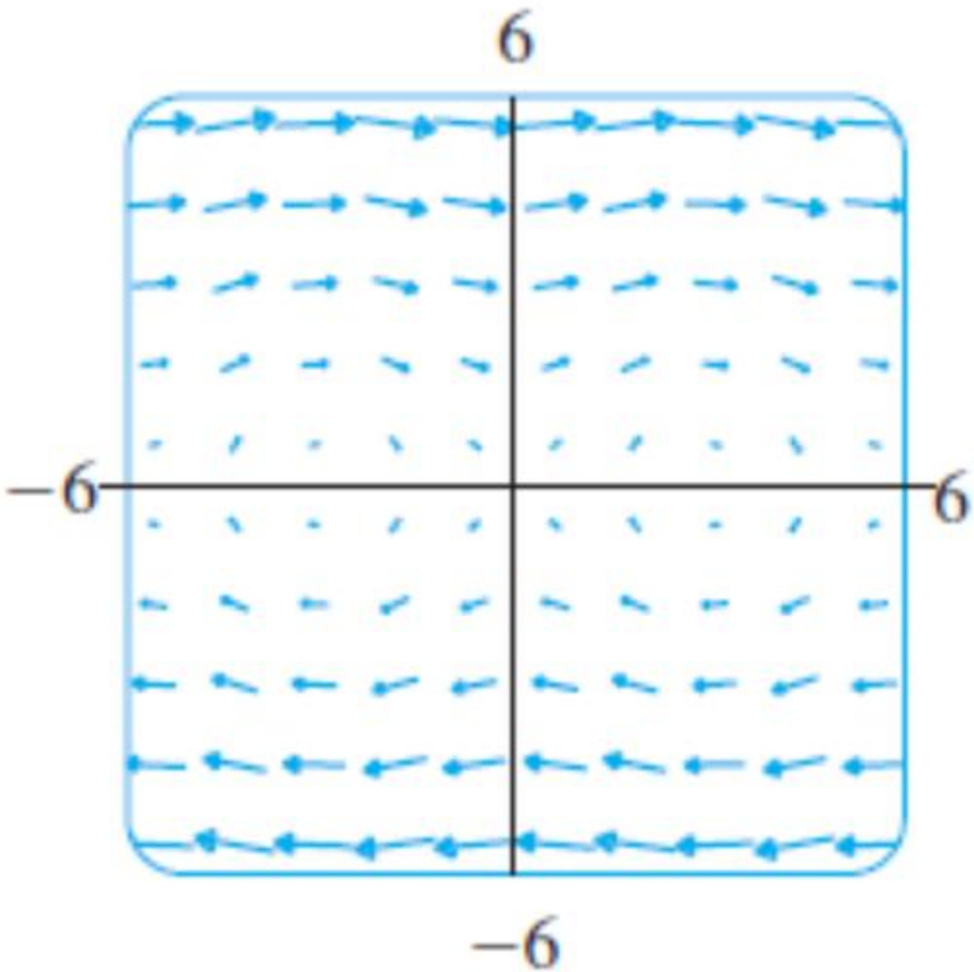
1. What is a vector field?

$$\mathbf{F}(x, y) = \langle -y, x \rangle$$



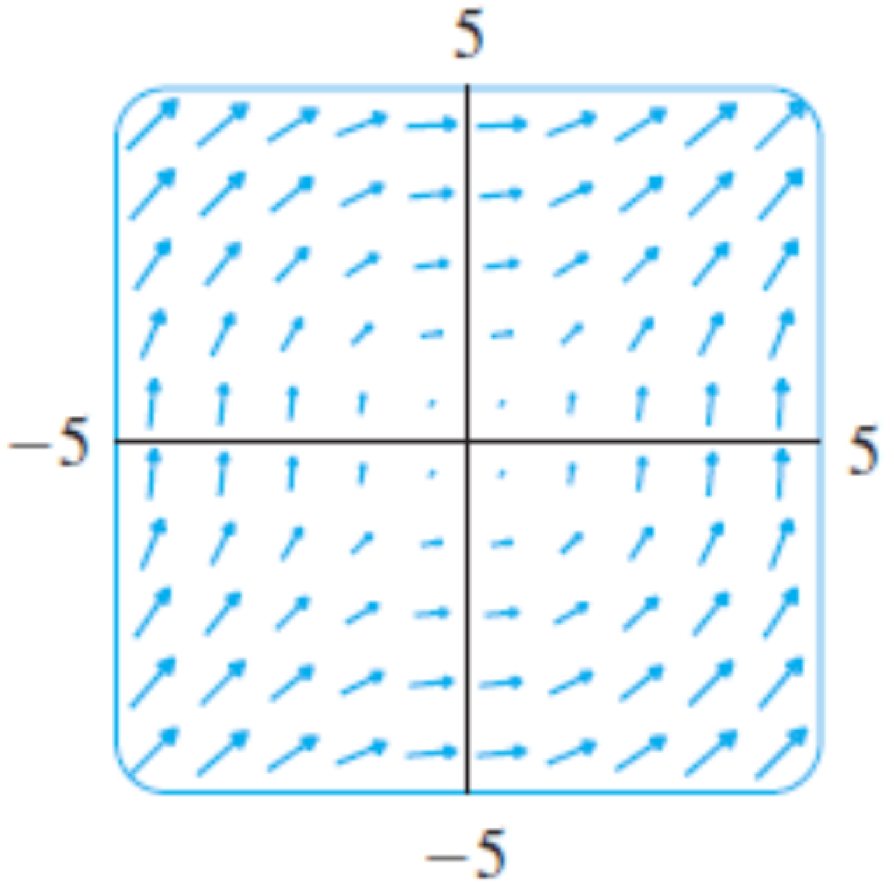
1. What is a vector field?

$$\mathbf{F}(x, y) = \langle y, \sin x \rangle$$



1. What is a vector field?

$$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$$



1. What is a vector field?

Def:

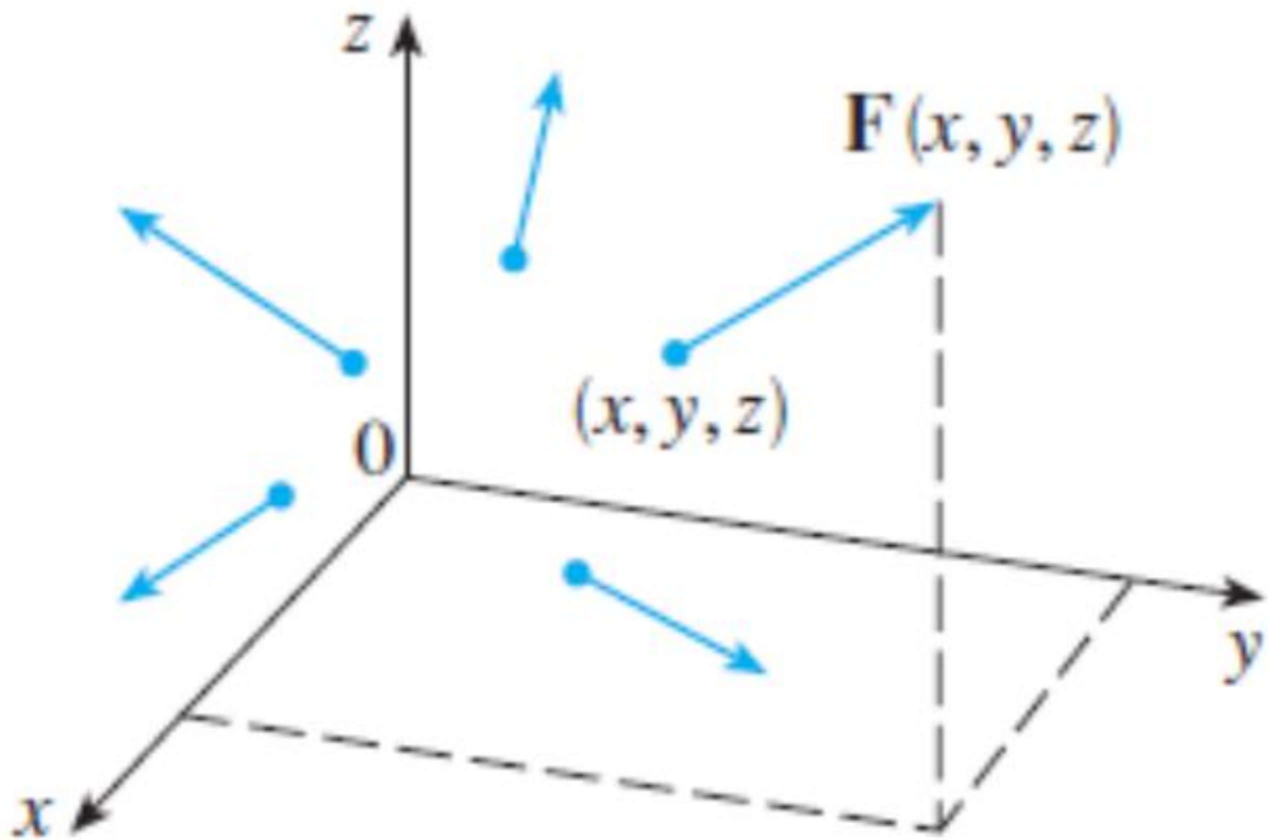
2. Let E be a set of points in $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\vec{F}(x, y, z)$.
   That is, $\vec{F}: E \subset \mathbb{R}^3 \to V_3$.

Notes:

- We draw a vector field on $\mathbb{R}^3$ the same way as for those on $\mathbb{R}^2$. Just now everything is done in space instead of in the plane.

1. What is a vector field?

Vector field on $\mathbb{R}^3$



1. What is a vector field?

<u>Def</u>:

2. Let E be a set of points in $\mathbb{R}^3$. A <u>vector field</u> on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\vec{F}(x, y, z)$.
   That is, $\vec{F}: E \subset \mathbb{R}^3 \rightarrow V_3$.

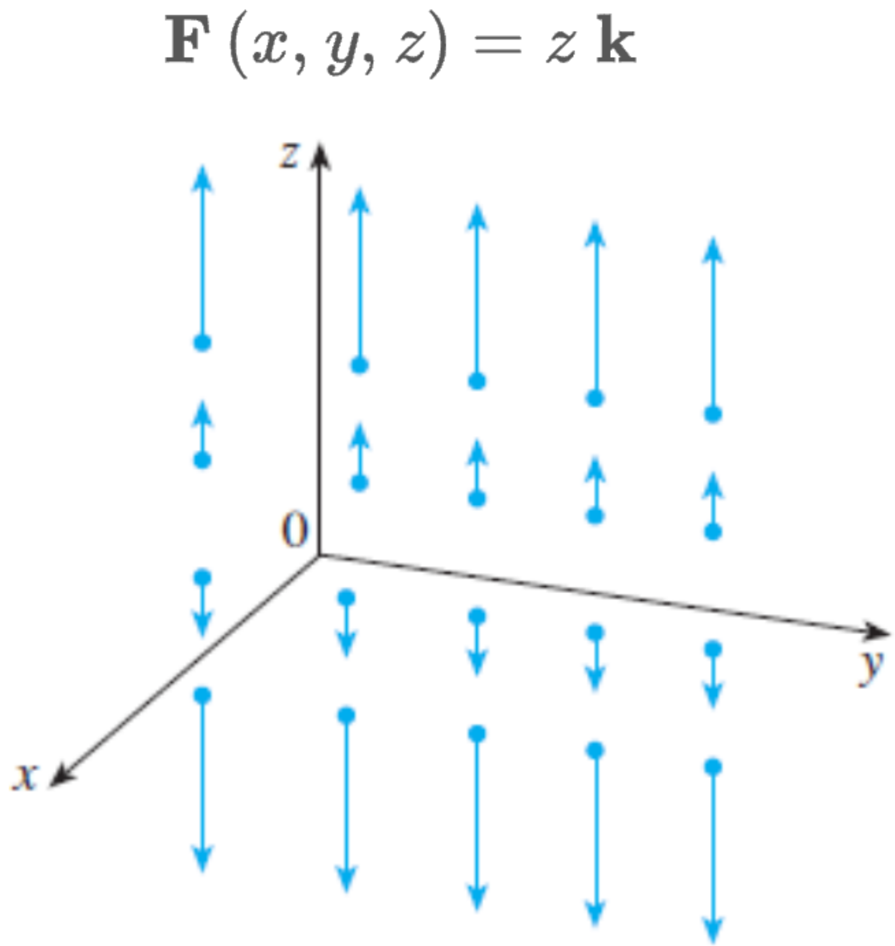
<u>Notes</u>:

- A 3-variable function from E to $\mathbb{R}$ is also called a <u>scalar function</u>. We've seen these before as well.
- If P, Q, and R are to scalar functions, then a vector field on $\mathbb{R}^3$ is the function $\vec{F}$ defined by

$$\vec{F}(x, y, z) =< P, Q, R > \text{ or } \vec{F}(x, y, z) = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$$

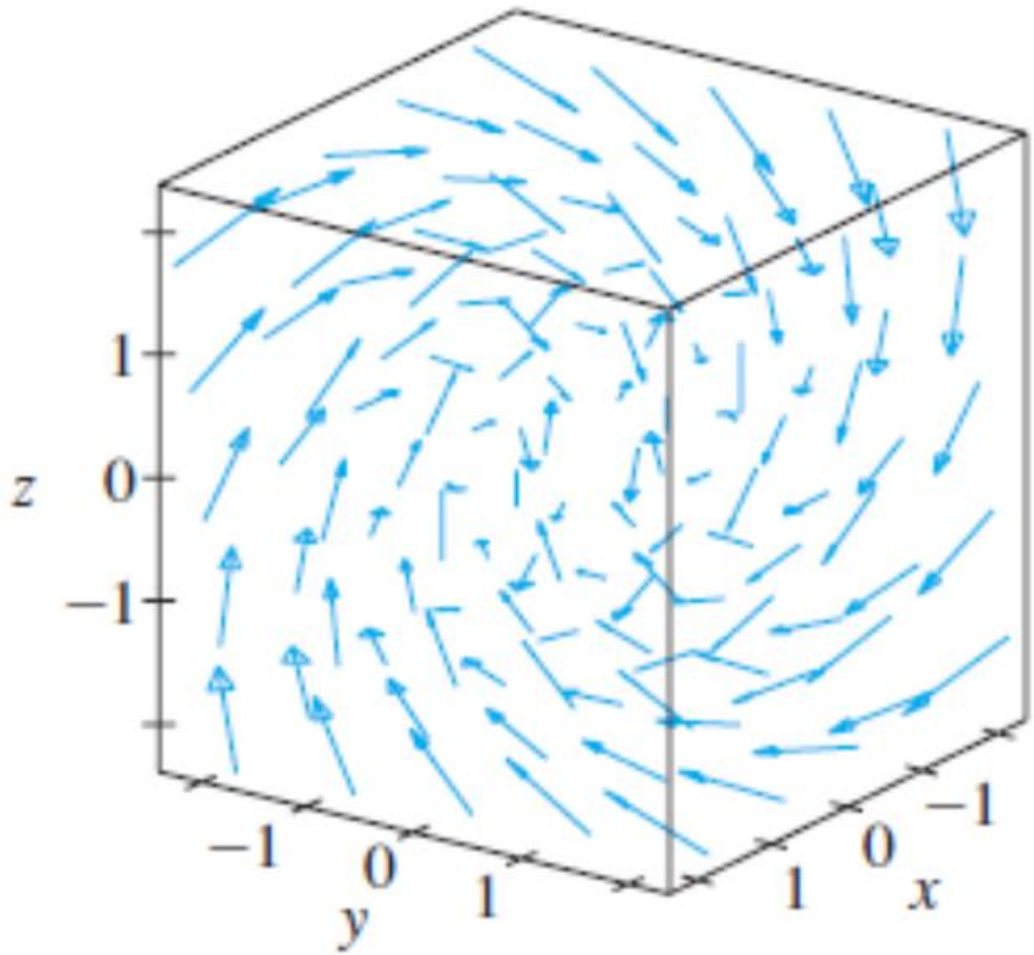
1. What is a vector field?

<u>Ex 2</u>: Sketch the vector field on $\mathbb{R}^3$ given by $\vec{F}(x, y, z) = z\vec{k}$.



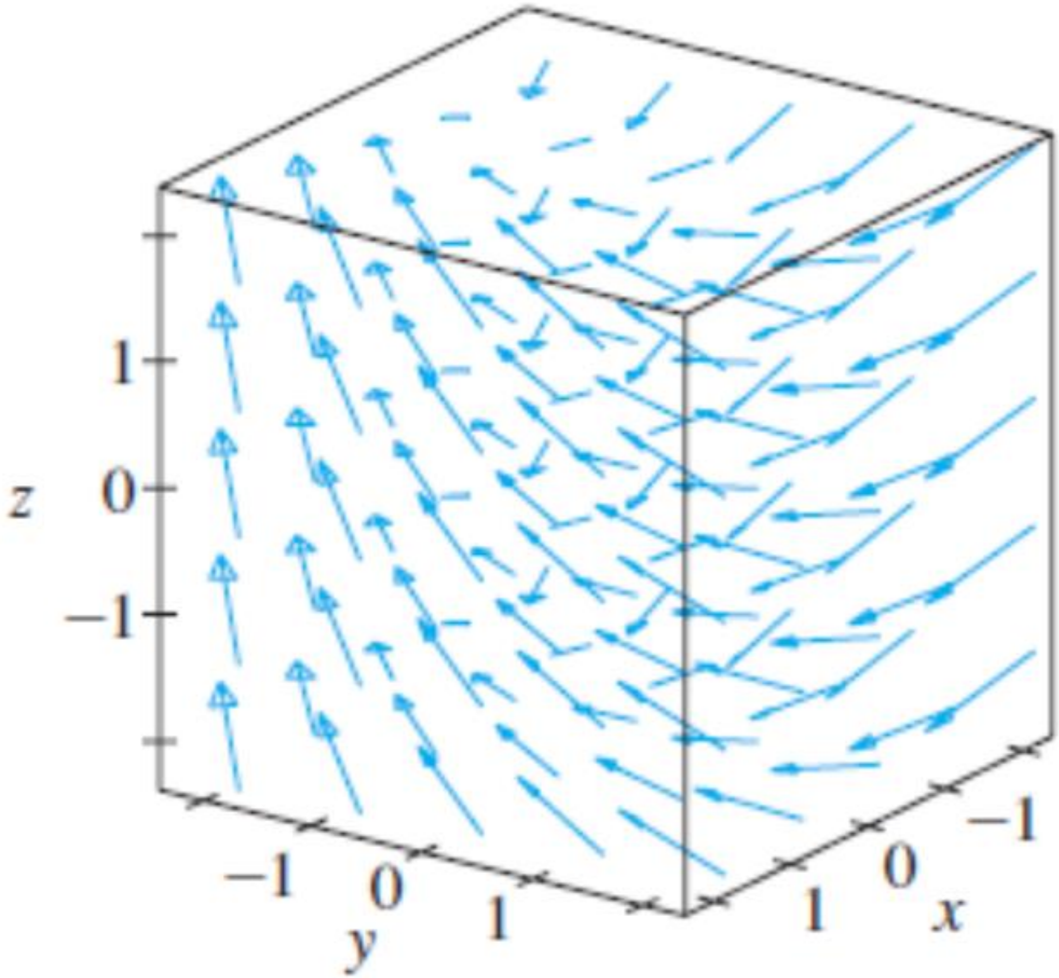
1. What is a vector field?

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$



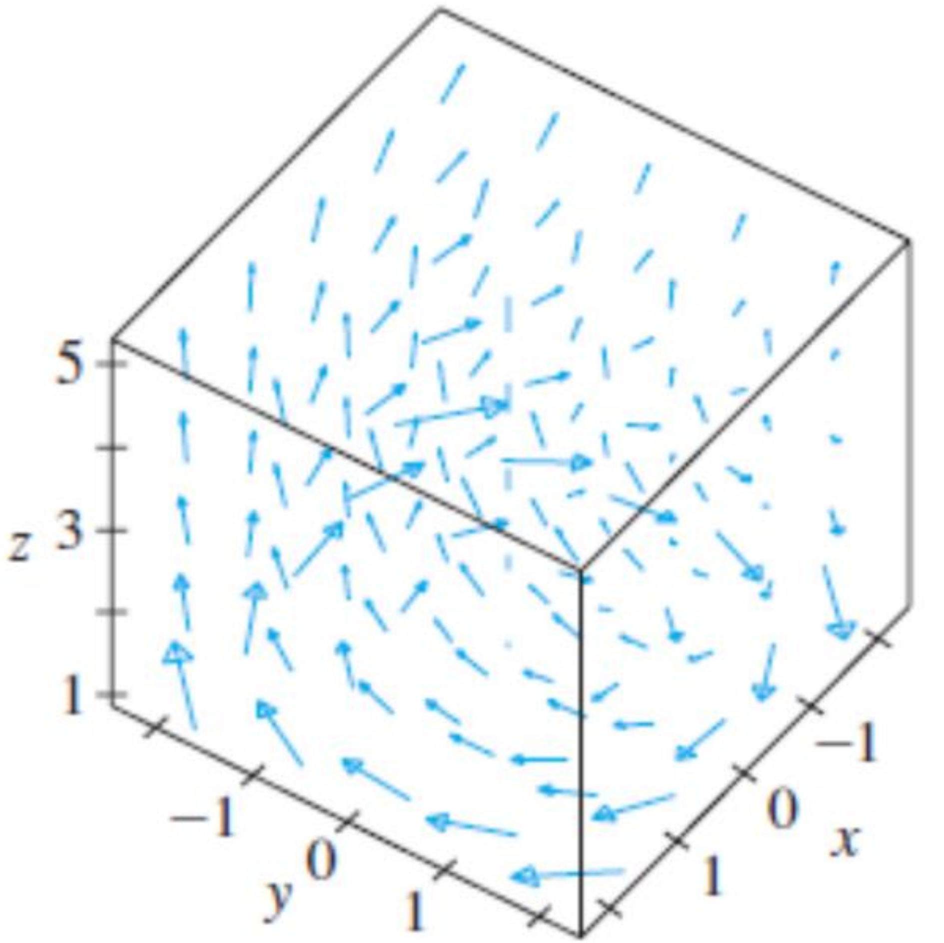
1. What is a vector field?

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$$



1. What is a vector field?

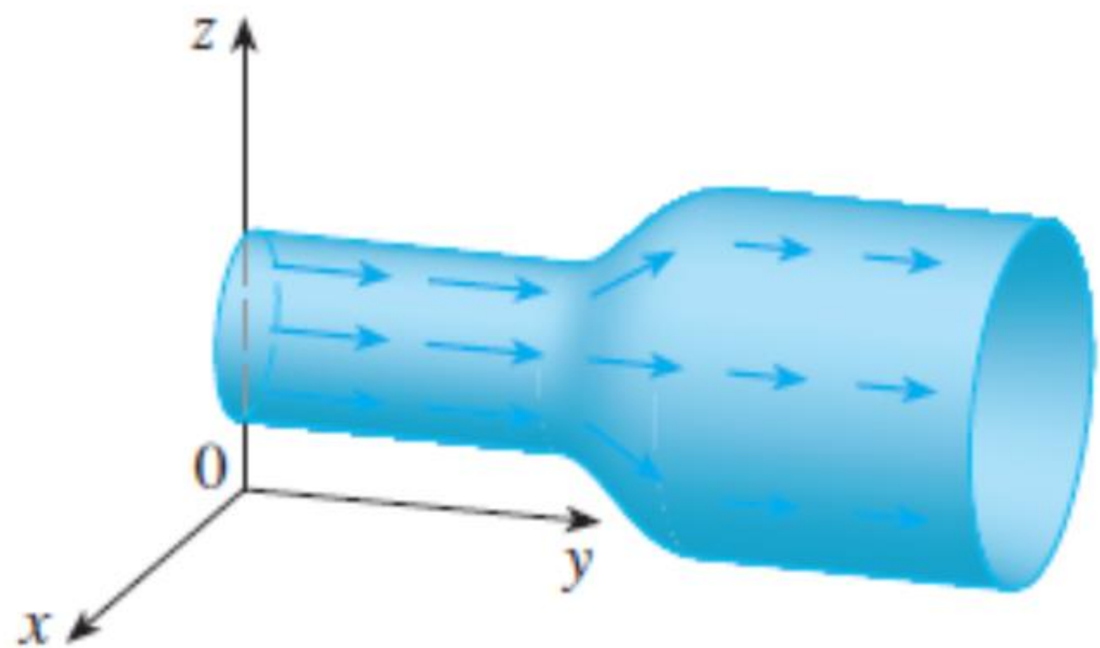
$$\mathbf{F}(x, y, z) = \frac{y}{z}\,\mathbf{i} - \frac{x}{y}\,\mathbf{j} + \frac{z}{4}\,\mathbf{k}$$



1. What is a vector field?

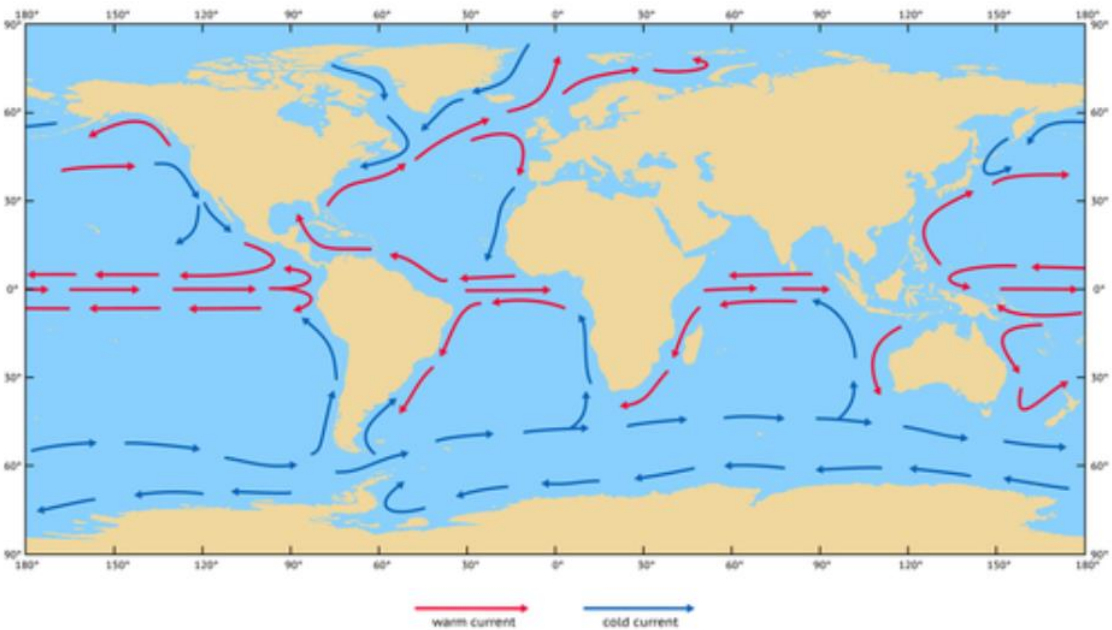
<u>Ex 3</u>: Water flowing through a pipe. Let the vector field be $\vec{V}(x, y, z)$, the water's velocity vector field.

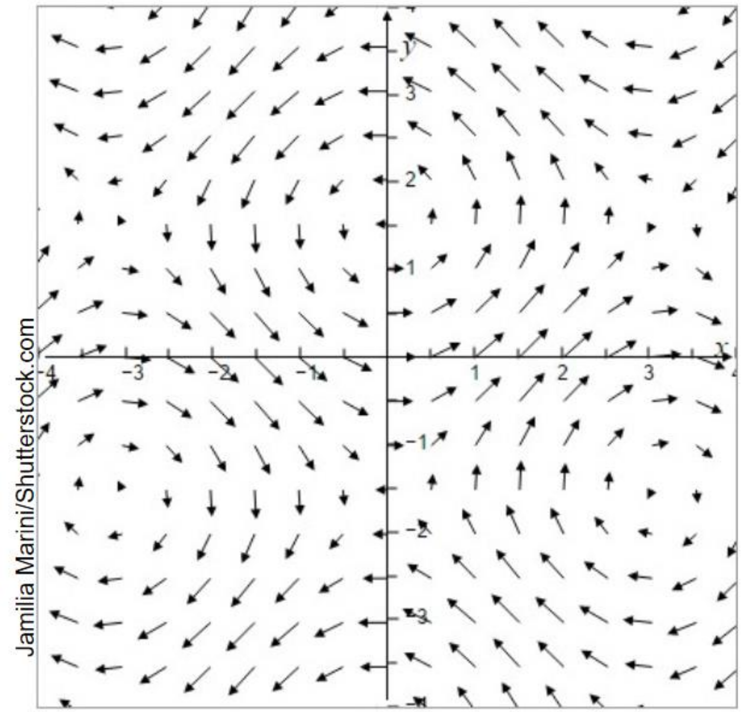
Velocity field in fluid flow



1. What is a vector field?

Ocean currents…





1. What is a vector field?

Ex 4: Newton's law of gravitation between 2 objects of mass M and m. Assume an object of mass M is placed at the origin in $\mathbb{R}^3$ and another object of mass m could be at any other point in space. We can map out a force field…the vector field that at each location in space tells me the force the object of mass m will experience if it is at that point.

1. What is a vector field?

Ex 4: Newton's law of gravitation force field…

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

$$\mathbf{F}(x, y, z) =$$

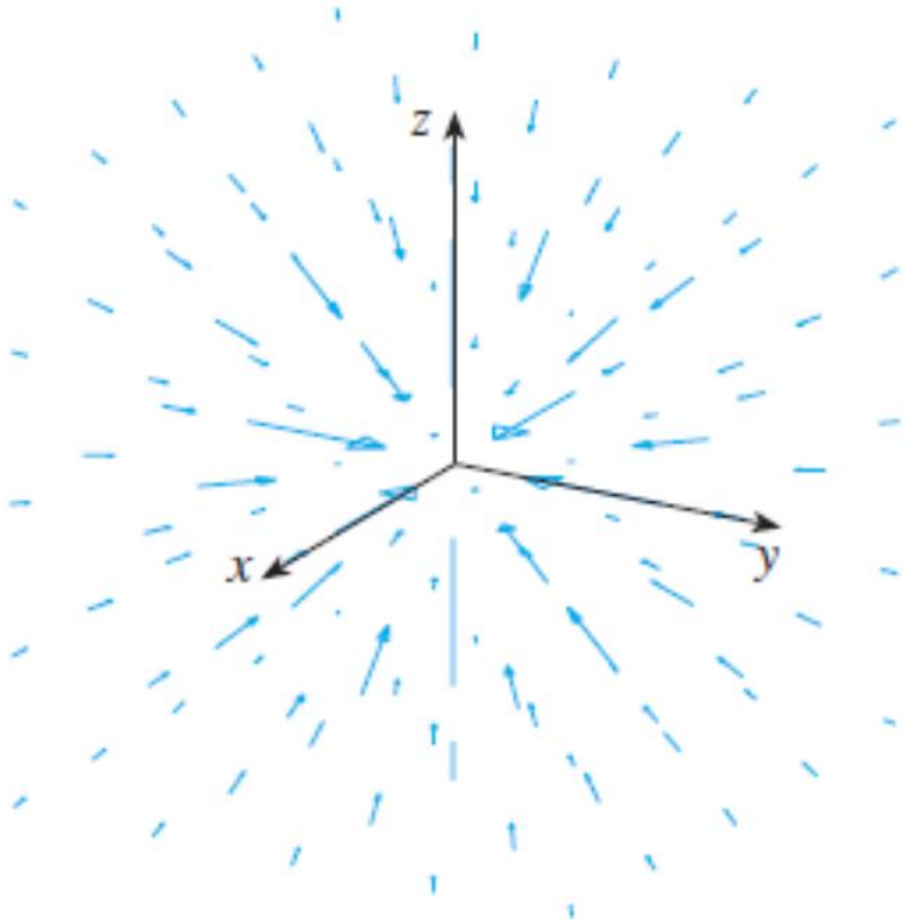
$$\frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

Derive?

1. What is a vector field?

Ex 4: Newton's law of gravitation force field…

Gravitational force field



1. What is a vector field?

Ex 5: Coulombs law force between 2 objects of charge Q and q. Assume an object with charge Q is placed at the origin in $\mathbb{R}^3$ and another object with charge q could be at any other point in space. We can map out a force field…the vector field that at each location in space tells me the force the object with charge q will experience if it is at that point.

A more common field is the electric field E which is the force that an object with charge 1 Coulomb will experience if placed at a given point in space.

1. What is a vector field?

Ex 5: Coulombs law force between 2 objects of charge Q and q.

$$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\mathbf{x}$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

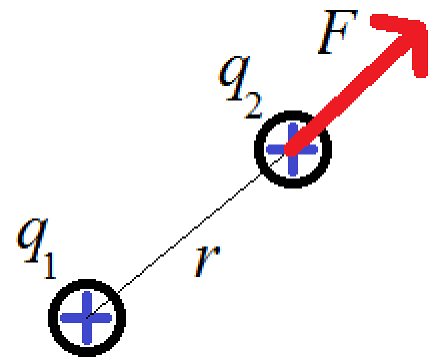
Coulomb’s Law and Electric Fields

Coulomb's Law

Suppose 2 point charges are a distance r apart. Suppose one point charge has charge q_1 and the other has charge q_2. Then the magnitude of the force experienced by either charge due to the other is

$$F = \frac{k|q_1||q_2|}{r^2}$$

$$k = 9 \times 10^9 \frac{N \cdot m^2}{C^2}$$



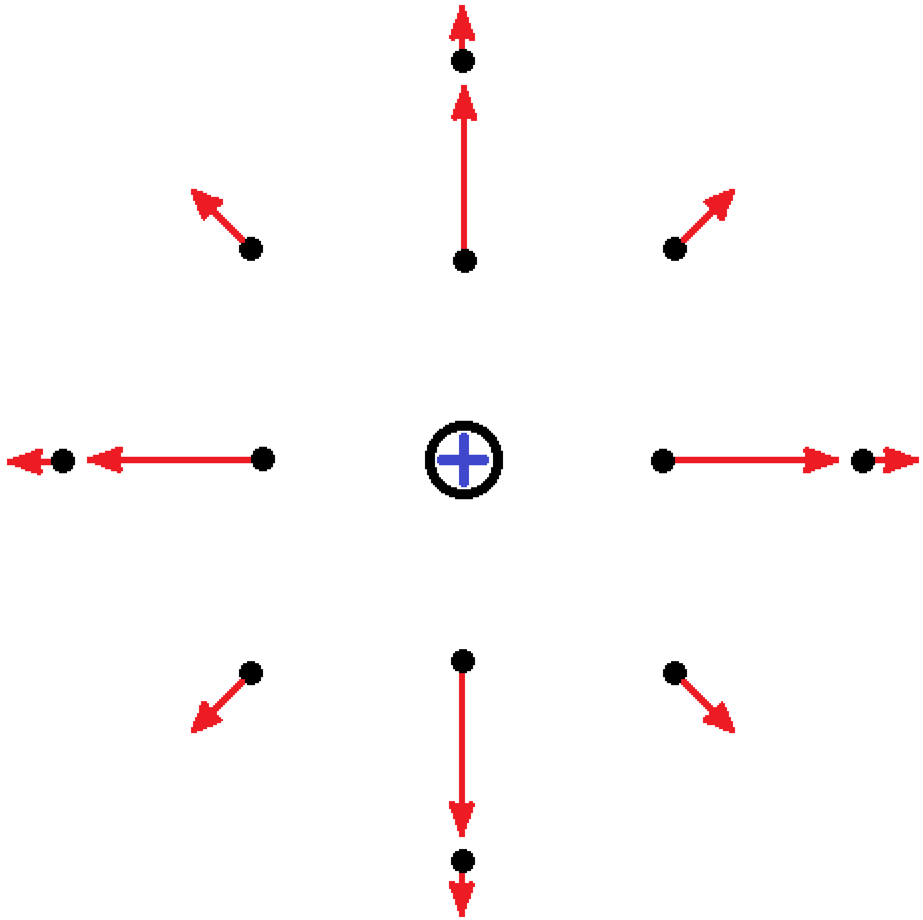
Direction of force is determined by…

1. Like charges repel
2. Unlike charges attract
3. Direction of the force is along the line connecting the 2 charges

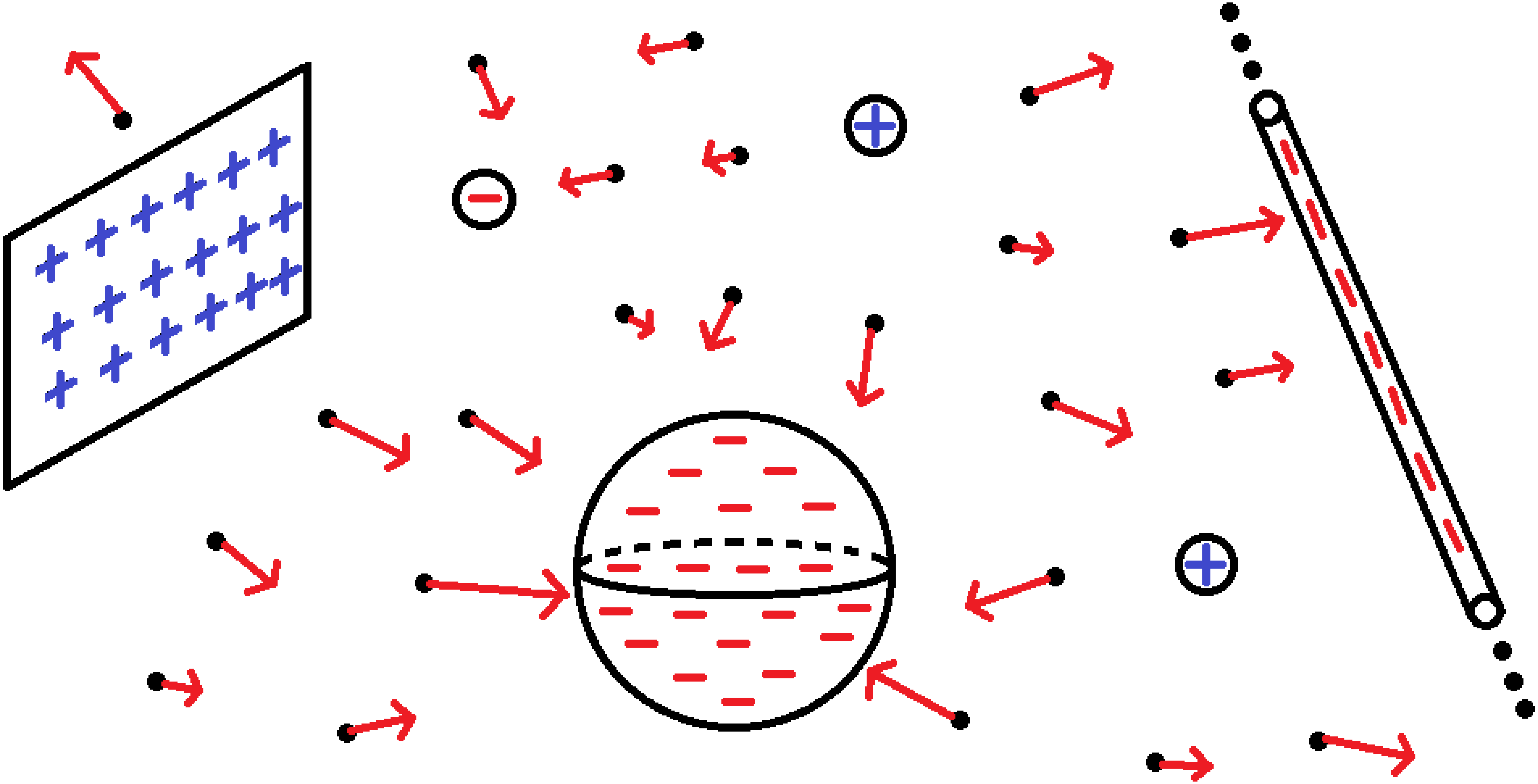
Electric Field

Suppose you have a charge distribution (nailed down in place). If you bring a +1 C charge to a given point in space, it will experience a force due to the charge distribution. There is such a force vector for each point in space. The collection of all such force vectors is the electric field of the charge distribution.

Electric Field example 1: Electric field due to a single point charge



Electric Field example 2: Electric field from a charge distribution



2. The gradient vector field

Given a 2 variable scalar function $f(x, y) \quad (f: \mathbb{R}^2 \to \mathbb{R})$,
its gradient $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ is a vector field on $\mathbb{R}^2$ $(\nabla f: \mathbb{R}^2 \to V_2)$.

Given a 3 variable scalar function $f(x, y, z) \quad (f: \mathbb{R}^3 \to \mathbb{R})$,
its gradient $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ is a vector field on $\mathbb{R}^3$ $(\nabla f: \mathbb{R}^3 \to V_3)$.

3. Conservative vector fields

Def:

- A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some scalar function.
- That is, $\vec{F}$ is conservative if there is a scalar function f such that $\nabla f = \vec{F}$.
- In this situation f is called a potential function for $\vec{F}$.

2. The gradient vector field

Ex 7: If $f(x, y, z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$, then

$$\nabla f(x, y, z) = \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}\hat{\imath} + \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}\hat{\jmath} + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

which is the gravitational vector field from example 4. So the gravitational vector field from example 4 is a conservative vector field.